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Partial Control of Linear Inventory Systems

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Feedback and feedforward inventory control systems are derived which give improved probability of holding all inventories between predetermined limits, even when there are fewer controllers than controlled variables. The concept of partial controllability is developed to describe systems with uncontrollable states. An example shows that four tank levels can be held within limits with high probability, with only one manipulated flow rate used.

Sometimes one wishes to regulate a system over which control can be exercised only partially. In a *partially controllable* system, the controlled variables cannot all be driven to their respective set points simultaneously after a disturbance, although they can all be influenced in some way by the manipulated variables. The concept of partial controllability depends on the way the manipulated and controlled variables interact rather than on the nature of the disturbances, be they random or deterministic. Quality of performance of a partially controlled system will, however, depend on the character of the disturbances. A well-designed control scheme gives the best performance possible no matter what sort of external upsets perturb the system, although the quality of this best behavior depends on how badly the disturbances fluctuate.

For example, consider the process in Figure 1. Two substances are delivered, more or less at random, to their respective feed tanks and subsequently pumped to a chemical reactor in constant (say equal) proportions. The reacted mixture is fractionated in a fixed ratio (5 to 1 in this case) into two products which are pumped to separate tanks until ordered by customers who are not entirely predictable. If the flow of mixture through the reactor is manipulated, the levels in all four tanks are influenced. Yet one cannot hold the inventories simultaneously constant at predetermined levels; if one level is fixed, the others must fluctuate uncontrolled. This system is only partially under control.

If the level in only one tank were important, the system would be completely controllable, and a controller could be designed to hold the level constant. Or if the four flows in and out of the tanks did not need to be held in fixed ratios and therefore could be manipulated independently of each other, four controllers could be used to hold each level constant. This system would also be completely controllable. Although powerful design methods can be brought to bear on such completely controllable systems, they do not apply to the partially controllable process of Figure 1. Thus in the present state of technology one finds that such processes are left to the manual regulation of an operator or a manager rather than controlled automatically.

How is it then that an operator can control a partially controllable system when control engineers cannot? The answer is in the interpretation of the word *control.* Control theorists say that a variable is controlled only if it can be driven to some predetermined set point in finite time. Common English usage of the word gives more leeway, however, for the dictionary meaning of the verb *control* is to keep within limits (*4*). Employing the latter definition, an operator does not care what the levels are as long as no tank empties or overflows, and he manipulates the reactor flow to avoid these unsatisfactory circumstances as long as he can. This article shows how to perform such adjustments automatically and, in a certain sense, optimally, even though the system variables are not considered controllable by many control engineers.

Kalman, Ho, and Narendra (*1*), in constructing a rigorous mathematical foundation for control theory, make precisely the distinction needed to resolve this difficulty. They point out that one should not speak of the controllability of systems, but only of the controllability of certain states (or phases) depending on the directly measured controlled variables. Thus the semantic difficulty can be resolved by referring to a variable as *partially controllable* whenever it is the resultant of uncontrollable as well as

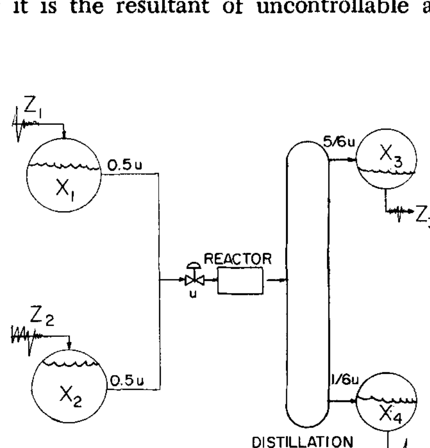

Fig. 1. System flow diagram.

controllable variables. The term *uncontrollable* can then be reserved for variables totally unaffected by the manipulated variables. Just as a system having all its variables (un)controllable is said to be a(n) (un)controllable system, so a system with any partially controllable variables is described as partially controllable. As a final point of terminology, let it be noted that the overdetermined systems (*3, 5, 6*) studied in the past are, in the sense of this article, partially controllable, the former term being abandoned as confusing and not as compatible with modern control theory as the latter.

This article develops a feedforward and feedback design theory for certain partially controllable linear systems arising in production and inventory scheduling problems. It uses well-known properties of linear vector spaces to apply the theory of Kalman, Ho, and Narendra to certain types of partially controllable storage systems. Controlled variables are shown to be expressible as linear combinations of controllable variables and uncontrollable variables, which can be found by methods given. As in the previous theory of overdetermined systems, operations are said to be satisfactory whenever all controlled variables are within their predetermined limits. If disturbances are deterministic and known, the theory shows how to keep operation satisfactory where possible. When upsets are random, control effectiveness is measured by probability of satisfactory operation at some future time, given the initial state of the system. Methods for estimating the maximum attainable probability as well as feedback synthesis techniques for achieving the optimum are developed, as well as a method for estimating the maximum attainable probability in the industrially important special case of no self-regulation. There is usually more than one control system giving optimum probability, and it is shown how to choose one which also minimizes the time to correct for upsets when the manipulated variables are bounded. This maximum probability minimum time controller switches the manipulated variable between extremes. Fully linear control systems are also described. Computations and synthesis are demonstrated for the process in Figure 1, which is a prototype of systems with storage depots in parallel. The series arrangement having been solved previously (*6*), arbitrary series-parallel arrangements of storage depots may now be placed under improved probability partial control.

PROBLEM DEFINITION

A linear dynamical control system with constant coefficients can be described by the matrix differential equation:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \tag{1}$$

In many inventory control problems the states $\mathbf{x}$ represent quantities stored, the manipulated variables $\mathbf{u}$ are flow rates, and the matrix $\mathbf{F}$ has a special character to be exploited in this article. Consider first a tank i containing a liquid which exerts pressure on the pump removing material from the vessel and therefore increasing the flow rate. Under these circumstances the tank is said to possess self-regulation, and the i^{th} diagonal element f_{ii} of $\mathbf{F}$ will be negative. In a tank without this influence of quantity stored upon exit flow, there is no self-regulation, and $f_{ii} = 0$. In general

$$f_{ii} \leq 0; \ i = 1, \ldots, n \tag{2}$$

Next imagine that the exit pipe from tank i is connected below the liquid level in tank j so that back pressure from the latter tank slows down the flow from the former. Then $f_{ij} > 0$, and since the influence is mutual, that is, the head in tank i increases flow into tank j, $\mathbf{F}$ must be symmetric:

$$f_{ij} = f_{ji} > 0 \tag{3}$$

and

$$\mathbf{F} = \mathbf{F}' \tag{4}$$

Now any symmetric matrix $\mathbf{F}$ can be written $\mathbf{P'DP}$, where $\mathbf{D}$ is the real diagonal matrix of eigenvalues of $\mathbf{F}$, and $\mathbf{P}$ is a normal orthogonal matrix whose rows are the eigenvectors of $\mathbf{F}$. Hence the orthogonal transformation $\xi = \mathbf{P}\mathbf{x}$ changes Equation (1) into

$$\dot{\xi} = \mathbf{D}\xi + \mathbf{PGu}$$

No generality is lost in assuming that this transformation has been carried out, or, equivalently, that $\mathbf{F}$ is diagonal in the first place. If the system is to be stable, the diagonal elements must be negative as in Equation (2).

Kalman, Ho, and Narendra (*1*) have shown that if all the diagonal elements are nonzero and distinct, then under certain circumstances the system is completely controllable, a case which, while certainly arising in practice, is of no concern in this article on partial controllability. It will be assumed that all diagonal elements are the same; that is

$$\mathbf{F} = -\alpha\mathbf{I} \tag{5}$$

where $\mathbf{I}$ is the n by n unit matrix, and α is non-negative. When $\alpha > 0$, every tank has self-regulation, while $\alpha = 0$ indicates that there is no self-regulation anywhere, the more usual situation in industry. Thus Equation (1) becomes

$$\dot{\mathbf{x}} = -\alpha\mathbf{x} + \mathbf{G}\mathbf{u} \tag{6}$$

For any α the columns of the matrices $\mathbf{FG}$, $\mathbf{F}^2\mathbf{G}$, . . . , $\mathbf{F}^{n-1}\mathbf{G}$ are linear combinations of those of $\mathbf{G}$, which implies that the rank of the composite matrix ($\mathbf{G}$, $\mathbf{FG}$, $\mathbf{F}^2\mathbf{G}$, . . . , $\mathbf{F}^{n-1}\mathbf{G}$) is equal to the rank of $\mathbf{G}$, which is the number of controllers m. By theorem 10 of Kalman, Ho, and Narendra (*1*), this implies that these inventory systems are not completely controllable unless $m = n$, rarely the case in practice. Assume that the columns of $\mathbf{g}_h$ of G are normal and orthogonal

$$\mathbf{g}_h'\mathbf{g}_i = \delta_{hi} \ \text{(Kroneckers' delta)} \tag{7}$$

As demonstrated in the example, this simplifying assumption is not restrictive. The control and state variables are assumed to be bounded above and below by constants, the control bounds representing limits on control effort and the state bounds being dictated by the capacities of the storage depots:

$$\mathbf{u}^- \leq \mathbf{u} \leq \mathbf{u}^+ \tag{8}$$

$$\mathbf{x}^- \leq \mathbf{x} \leq \mathbf{x}^+ \tag{9}$$

The initial conditions on all states $\mathbf{x}$ are, without loss of generality, taken to be zero.

The random disturbances $\mathbf{z}(t)$ occurring during a time interval of duration T may be accounted for by incorporating them into the constant of integration of the integrated form of Equation (6)

$$\mathbf{x}(T) = \int_0^T \exp\left[-\alpha(T-t)\right]\dot{\mathbf{z}}(t)\,dt + \mathbf{G}\int_0^T \exp\left[-\alpha(T-t)\right]\mathbf{u}(t)\,dt \tag{10}$$

which simplifies for systems without self-regulation to

$$x(T) = \mathbf{z}(T) + \mathbf{G}\int_0^T \mathbf{u}(t)\,dt \tag{11}$$

A simple linear change of variable will make the expected values of all random variables zero; assume this has been done (*6*). Satisfactory operation at time T is defined as having all inequalities (9) satisfied then. The probability of satisfactory operation at time T

$$p(T) = Pr\{\mathbf{x}^- \leq \mathbf{x}(T) \leq \mathbf{x}^+\} \quad (12)$$

depends on the distribution of the random variables $\mathbf{z}$ as well as on how the controls $\mathbf{u}$ are manipulated in the interval $(0, T)$. Thus one wishes to find a feedback control $\mathbf{u}^*[\mathbf{x}(t)]$ maximizing the probability of satisfactory operation

$$p^*(T) = \max_{\mathbf{u}[\mathbf{x}(t)]} [p(T)] \quad (13)$$

The article shows how to synthesize a class of linear feedback control systems giving maximum probability of satisfactory operation, even when there are fewer controllers than controlled variables ($m < n$), and the system therefore is not completely controllable (*1*). Such systems, which arise often in inventory control problems, have been called *overdetermined* in past work (*3, 5, 6*). This term will be replaced by *partially controllable*.

ORTHOGONALIZATION

The problem is solved by finding, for the n dimensional vector space containing the states $\mathbf{x}$, a normal orthogonal basis upon which arbitrary states can be expressed as linear combinations of m controllable states $\mathbf{v}$ and n-m uncontrollable states $\mathbf{w}$. Since the controllable components can be adjusted at will, they can be used to construct error signals for feedback controllers. On the other hand, the uncontrollable components measure all the random fluctuations incapable of compensation by the control variables $\mathbf{u}$. Hence they can be used to compute the probability of satisfactory operation. The transformation partitions the effects of the random disturbances into two parts, one which can be annihilated by the controllers and another which cannot be diminished at all.

Let $\mathbf{j}_{m+1}, \ldots, \mathbf{j}_n$ be n-m normal n vectors, each orthogonal to the others, as well as to the columns $\mathbf{g}_1, \ldots, \mathbf{g}_m$ of $\mathbf{G}$. That is

$$\mathbf{j}_h'\mathbf{j}_i = \mathbf{j}_i'\mathbf{g}_h = 0 \quad (14)$$

Such vectors can always be found by the Gram-Schmidt orthogonalization procedure, although often in practice some of them can be found by inspection. The orthogonal set is not unique, and it is often helpful for the engineer to select vectors with simple physical significance, as illustrated in the example to follow. The vectors are collected into an n by n-m matrix $\mathbf{J}$:

$$\mathbf{J} \equiv (\mathbf{j}_{m+1}, \ldots, \mathbf{j}_n) \quad (15)$$

Define the m vectors $\mathbf{v}$ and the $(n-m)$ vectors $\mathbf{w}$ by

$$\mathbf{v} \equiv \mathbf{G}'\mathbf{x} \quad (16)$$

$$\mathbf{w} \equiv \mathbf{J}'\mathbf{x} \quad (17)$$

From Equations (16) and (17) one gets

$$[\mathbf{G}, \mathbf{J}] \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \mathbf{G}\mathbf{v} + \mathbf{J}\mathbf{w} = \mathbf{x} \quad (18)$$

Equations (5), (7), (14), (16), and (18) give, upon differentiation with respect to time

$$\dot{\mathbf{v}} = -\alpha\mathbf{v} + \mathbf{u} \quad (19)$$

which means that each phase of $\mathbf{v}$ can be influenced directly by exactly one of the control variables $\mathbf{u}$. For this reason the elements of $\mathbf{v}$ are called the *controllable phases*. Similarly, using Equation (18) instead of (17) one gets

$$\dot{\mathbf{w}} = -\alpha\mathbf{w} \quad (20)$$

which means that the *uncontrollable phases* $\mathbf{w}$ cannot be adjusted at all. The interpretation of $\mathbf{v}$ as controllable and $\mathbf{w}$ as uncontrollable is even more clear when there is no self-regulation ($\mathbf{F} = 0$), for then

$$\dot{\mathbf{v}} = \mathbf{u} \quad (21)$$

and

$$\dot{\mathbf{w}} = 0 \quad (22)$$

as in the example at the end of the article. Thus the transformation partitions every state into controllable and uncontrollable phases as in reference *1*.

CONTROLLER SYNTHESIS

Next it will be shown that the transformation when applied to the inequalities (9) yields simple inequalities on the components of $\mathbf{v}$ and $\mathbf{w}$ which are satisfied if the system is operating satisfactorily. Let K be the number of square m by m nonsingular submatrices $\mathbf{G}_k$ ($k = 1, \ldots, K$) constructible from the rows of $\mathbf{G}$ and let g_{ihk} be the element in row i and column h in $\mathbf{G}_k^{-1}$, the inverse matrix of any $\mathbf{G}_k$. Define

$$X_{hik}^+ \equiv \begin{cases} x_h^+ & \text{if } g_{ihk} > 0 \text{ or } j_{ih} > 0 \\ x_h^- & \text{if } g_{ihk} < 0 \text{ or } j_h < 0 \end{cases} \quad (23a)$$

and

$$X_{hik}^- \equiv \begin{cases} x_h^- & \text{if } g_{ihk} > 0 \text{ or } j_{ih} > 0 \\ x_h^+ & \text{if } g_{ihk} < 0 \text{ or } j_{ih} < 0 \end{cases} \quad (23b)$$

Let these elements be assembled into m vectors $\mathbf{X}_k^+$ and $\mathbf{X}_k^-$, respectively, and define

$$\mathbf{v}_k^+ \equiv \mathbf{G}_k^{-1}\mathbf{X}_k^+ \quad (24a)$$

$$\mathbf{v}_k^- \equiv \mathbf{G}_k^{-1}\mathbf{X}_k^- \quad (24b)$$

as well as the n vectors $\mathbf{X}^+$ and $\mathbf{X}^-$ and

$$\mathbf{w}^+ \equiv \mathbf{J}'\mathbf{X}^+ \quad (25)$$

$$\mathbf{w}^- \equiv \mathbf{J}'\mathbf{X}^- \quad (26)$$

Assemble those row vectors of $\mathbf{J}$ having the same indices of $\mathbf{G}_k$ into an m by $(n-m)$ matrix $\mathbf{J}_k$, and define

$$\mathbf{V}_k^+(\mathbf{x}) \equiv \mathbf{v}_k^+ - \mathbf{G}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x} \quad (27)$$

$$\mathbf{V}_k^-(\mathbf{x}) \equiv \mathbf{v}_k^- - \mathbf{G}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x} \quad (28)$$

Now consider the vector inequalities

$$\mathbf{V}_k^-(\mathbf{x}) \leq \mathbf{v}_k \leq \mathbf{V}_k^+(\mathbf{x}); \quad k = 1, \ldots, K \quad (29)$$

$$\mathbf{w}^- \leq \mathbf{w} \leq \mathbf{w}^+ \quad (30)$$

It is contended that inequalities (29) and (30) are satisfied if inequalities (9) are also satisfied; that is, when the system is operating satisfactorily. To see why this is so, multiply the i^{th} component of inequality (9) by the constant j_{hi}. If $j_{hi} > 0$, then

$$j_{hi}\, x_i^- \leq j_{hi}\, x_i \leq j_{hi}\, x_i^+$$

But if $j_{hi} < 0$, then

$$j_{hi}\, x_i^+ \leq j_{hi}\, x_i \leq j_{hi}\, x_i^-$$

When these inequalities are summed over i running from 1 to n, and Equations (23), (25), and (26) are applied, one obtains

$$w_h^- \leq w_h \leq w_h^+$$

which is a typical component of inequality (30). Next combine (9), (17), and (18) to obtain

$$\mathbf{x}_k \leq \mathbf{G}_k\,\mathbf{v} + \mathbf{J}_k\mathbf{J}'\mathbf{x} \leq \mathbf{x}_k^+$$

and multiply throughout by $\mathbf{G}_k^{-1}$. Definitions (23), (24), (27), and (28) guarantee that the senses of the vector inequalities are preserved, so that transposition of $\mathbf{J}_k\mathbf{J}'\mathbf{x}$ is all that is needed to establish inequalities (29).

When the converse is true, that is, when satisfaction of Equations (29) and (30) implies satisfactory operation, one can obtain maximum probability of satisfactory operation simply by computing all $V_k^-(x)$ and $V_k^+(x)$ and by setting v within the most critical limits. For example, one might set each v_h according to $\frac{1}{2}\left[\min_k V_{hk}^+(x) + \max_k V_{hk}^-(x)\right]$. The converse is true whenever the elements of G_k have the same signs as those of the corresponding elements of G_k^{-1} for all k. This is the case when there is only one manipulated variable, as in the example, as well as when all tanks are in series, a situation described previously (*6*). When the converse does not hold, one can still use this mode of control, although one cannot be sure it is optimal.

A simpler method of control would be to set v at a constant value, say at the midpoint of the initial ranges when x = 0. In this case

$$v_h(0) = \frac{1}{2}\left[\min_k V_{hk}^+(0) + \max_k V_{hk}^-(0)\right]$$

Although this control scheme would rarely be optimal, it may give high enough probability of satisfactory operation to justify its use by its ease of application, as shown in the example. In either case, the vector v, which depends on x, can be subtracted from any set point $\bar{v}$ satisfying Equation (29) to give a vector of error signals e useful as inputs to the controllers:

$$e = \bar{v} - G'x \tag{31}$$

The controllers may be conventional linear three-action types, or if one wishes correction of upsets as soon as possible, switching or contactor controllers may be used in which

$$u_h = \begin{cases} u_h^+ & \text{if } e_h > 0 \\ 0 & \text{if } e_h = 0 \\ u_h^- & \text{if } e_h < 0 \end{cases} \tag{32}$$

Standard methods can be used to test a given control system for stability of the controllable v (*2*).

MAXIMUM PROBABILITY OF SATISFACTORY OPERATION

Although directions have been given for constructing feedback controllers, the fact that such controls give maximum probability of satisfactory operation remains to be proven. This will be done by showing that the uncontrollable w depends only on the random disturbances z and not on the control variables u. Hence one cannot do better than to keep the v within their limits; letting them drift outside leads only to immediate unsatisfactory operation. Therefore the control system described gives the maximum possible probability of satisfactory operation, provided it can react fast enough to disturbances to hold the v in control. Notice that this statement and its proof do not depend on knowledge of the probability distributions of the disturbances. Only the system performance, as measured by $p^*(T)$, is affected by the character of the disturbances—not the system optimality. Optimality is assured only when Equations (29) and (30) imply satisfactory operation, as in the special cases already noted.

Premultiplication of all terms of Equation (10) by J′ shows the exclusive dependence of w on z:

$$w = J'\int_0^T \exp[-\alpha(T-t)]\,\dot{z}(t)\,dt \tag{33}$$

When F = 0, this simplifies to

$$w = J'z(T) \tag{34}$$

so that if v is in control

$$Pr\{w^- \leq J'z(T) \leq w^+\} = p^*(T) \tag{35}$$

and one can calculate the optimum probability without working directly with w.

The disturbances z may be known rather than random. In this case probability of satisfactory operation is not a consideration, since operation will certainly be satisfactory if and only if for all T

$$w^- \leq J'\int_0^T \exp[-\alpha(T-t)]\,\dot{z}(t)\,dt \leq w^+ \tag{36}$$

When the noise vector can be measured, a feedforward control may be combined with feedback to give more rapid response to upsets. This is because Equations (17) and (36) give

$$J'x = w = J'z$$

even though $x \neq z$, J′ being singular. Substitution into inequalities (27) and (28) gives

$$V_k^+(z) = v_k^+ - G_k' J_k J' z$$

$$V_k^-(z) = v_k - G_k^{-1} J_k J' z$$

These functions of the upsets z may be used to construct error signals as for the feedback system described previously.

PARTIAL CONTROLLABILITY

The preceding developments call for modification of existing control terminology. If $m = n$, then there is no uncontrollable vector w, and every state x can be driven to zero. In the present literature (*1*) such systems are called *controllable*, a term no one can reasonably contest. On the other hand, any system with uncontrollable states may sometimes erroneously be dismissed as uncontrollable because it is not always possible to drive x to the origin. Yet in the sense of the verb control as defined in the dictionary, such systems may often be kept permanently under control by the methods of this article. This semantic paradox can be resolved by characterizing a system as partially controllable whenever there are any controllable components at all. In the past, partially controllable systems have been called *overdetermined* (*3, 5, 6*), but this terminology has proven confusing and will be abandoned.

EXAMPLE

Consider the hypothetical chemical process whose flow diagram is Figure 1. Equal amounts of raw materials 1 and 2 react to form two isomers 3 and 4, which are separated with a distillation column in the ratio 5 to 1 and pumped to product tanks. There is no self-regulation (F = 0). Since the control of a similar system by less effective methods has been described earlier (*3, 5*), derivation of the detailed material balance equations will be omitted. At the end of a 30-day planning period, the quantities in inventory are

$$x_1(30) = z_1(30) - \int_0^{30} \hat{u}\,dt$$

$$x_2(30) = z_2(30) - \int_0^{30} \hat{u}\,dt$$

$$x_3(30) = z_3(30) + (5/3)\int_0^{30} \hat{u}\,dt$$

$$x_4(30) = z_4(30) + (1/3)\int_0^{30} \hat{u}\,dt$$

where $\hat{u}$ is half the manipulated flow rate through the reactor. Equation (7) requires that the squares of the components of the control vector g_1 sum to unity, which is accomplished by choosing as the manipulated variable

$$u = \hat{u}\,[(-1)^2 + (-1)^2 + (5/3)^2 + (1/3)^2]^{1/2} = 2.21\hat{u}$$

Then

$$\mathbf{G}' = \mathbf{g}_1' = 0.452\ (-1, -1, 5/3, 1/3)$$

Suppose the bounds on the flow and inventories are

$$|u| \leq 100;\ |x_1| \leq 450;\ |x_2| \leq 750;\ |x_3| \leq 550$$

and

$$|x_4| \leq 250$$

By a coincidence generated to simplify the example the initial conditions fall exactly midway between the bounds, but in the general case the computations are only slightly more complicated. Let the random supplies and (negative) demands **z** have Gauss distributions with variances 10,000, 600,000, 250,000, and 10,000, respectively, all expected values being zero.

Two mutually orthogonal vectors $\mathbf{j}_2$ and $\mathbf{j}_3$, which are also orthogonal to $\mathbf{g}_1$, can be chosen by physical reasoning. Let $\mathbf{j}_2$ be proportional to the total amount of material in the system, certainly an uncontrollable quantity:

$$\mathbf{j}_2' = 0.500\ (1, 1, 1, 1)$$

Since materials 1 and 2 must be consumed in equal amounts, the difference $x_1 - x_2$ is also uncontrollable, so another normal orthogonal vector is

$$\mathbf{j}_3' = 0.707\ (1, -1, 0, 0)$$

The fourth vector is then uniquely determined by its required orthogonality and normality:

$$\mathbf{j}_4' = 0.213\ (-1, -1, -2, 4)$$

Hence a feedback controller can be constructed with error signal

$$e_1 = \bar{v}_1 - \mathbf{G}'\mathbf{x} = 0.452\,x_1 + 0.452\,x_2 - 0.753\,x_3 - 0.1506\,x_4$$

where $\bar{v}_1$ has been taken as zero. For this nonoptimal but simple control system the standard deviations of the four levels are, respectively, 205, 361, 258, and 115, which gives individual probabilities of satisfactory operation of 0.972, 0.962, 0.967, and 0.970, respectively. The true probability is bounded below by 0.877, the product of the four probabilities, and above by the least of them, 0.962. Hence

$$0.88 \leq p(30) \leq 0.96$$

with the true value being close to the upper bound because of correlation between the levels. If no control at all were used ($\mathbf{u} = 0$), which is often the control plan in practice, each tank must deal directly with the disturbance entering it, the standard deviations would be, respectively, 100, 400, 500, and 100, and the probability of satisfactory operation drops to 0.68. To achieve the same performance with no control as for the partially controlled system, one would have to increase the system volume 17%, mostly in tank 3, which must be expanded 94%.

To compute the probability of satisfactory operation for optimal control, one needs the bounds on the uncontrollable variables

$$|w_2| \leq 1000;\ |w_2| \leq 848;\ |w_3| \leq 700$$

and their standard deviations, respectively, 328, 292, and 179. The probability is 0.997, high enough so that the system volume would need to be increased almost 50% for nonoptimal partial control to achieve it. If this performance level is desired, and the additional computation needed is felt to be justified by the savings in new capacity, then one would generate the following four inequalities:

$$|v(\mathbf{x}) - 1.76x_1 + 0.452x_2 - 0.750x_3 - 0.150x_4| \leq 996$$
$$|v(\mathbf{x}) + 0.452x_1 - 1.76x_2 - 0.750x_3 - 0.150x_4| \leq 3390$$
$$|v(\mathbf{x}) - 0.452x_1 - 0.452x_2 - 0.576x_3 + 0.150x_4| \leq 733$$
$$|v(\mathbf{x}) - 0.452x_1 - 0.452x_2 + 0.750x_3 + 6.51x_4| \leq 1667$$

For any given **x** (say all of them 200), one would compute the most critical upper and lower bounds on **v**, in this case 554 (from the upper bound on x_1) and −737 (from the lower bound on x_3). Hence $\mathbf{v}(\mathbf{x}) = 1/2\ (554 - 737) = -92$. Notice that for this particular disturbance, the system would be satisfactory even if $v = 0$.

CONCLUSIONS

Methods have been presented for synthesizing feedback controllers giving maximum probability of satisfactory operation for partially controllable (formerly called *uncontrollable* or *overdetermined*) linear inventory systems. With this theory it is possible to mechanize the control of many complicated production and inventory control systems presently controlled either manually or not at all. Potential advantages are improved performance of existing systems or reduced investment in storage facilities in new ones.

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NOTATION

- $\mathbf{e}$ = m vector of error signals
- h, i = indices
- $\mathbf{g}_h$ = h^{th} column n vector of **G**
- $\mathbf{j}_h$ = h^{th} column n vector of **J**
- m = number of controllers
- n = number of state variables to be controlled
- **0** = null vector or null matrix
- $Pr(\cdot)$ = probability of ·
- $p(T)$ = probability of satisfactory operation at time T
- **u** = control m vector of manipulated variables
- **v** = m vector of set points
- **w** = $(n - m)$ vector of uncontrollable components
- **x** = n vector of state variables to be controlled
- **z** = n vector of random disturbances
- α = non-negative constant
- δ_{hi} = Kronecker's delta (= 1 when $h = i$ and = 0 when $h \neq i$)
- σ = standard deviation

Superscripts

- + (−) = upper (lower) bound
- * = optimum
- ′ = transposition

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